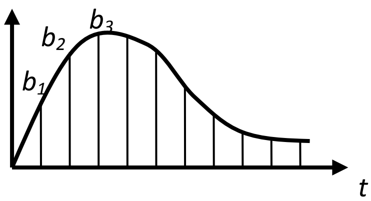
2.160 IDENTIFICATION, ESTIMATION, AND LEARNING

LECTURE NOTES NO. 14

15. Time-Series Data Compression

15.1 FIR Model

Finite Impulse Response Model



Consider a FIR Model with colored noise $v(t)$

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + \cdots\cdots + b_m u(t-m) + v(t) \tag{1}$$

Suppose that we want to identify the parameters involved in $G(q) = B(q)$ only,

$$\begin{aligned} \theta &= \begin{pmatrix} b_1 & b_2 & \cdots & b_m \end{pmatrix}^T \\ \varphi(t) &= \begin{pmatrix} u(t-1) & u(t-2) & \cdots & u(t-m) \end{pmatrix}^T \end{aligned} \tag{2}$$

Consider the least square estimate for a given set of data

$$Z^N = \{(u(t), y(t)) \mid t = 1 \cdots N\}$$

$$\hat{\theta}_N^{LS} = \arg\min_{\theta} \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} \left(y(t) - \phi^T(t)\theta \right)^2 \tag{3}$$

Let θ_0 be the vector of true parameter values. The parameter estimation error is given by

$$\hat{\theta}_N^{LS} - \theta_0 = \left(R(N) \right)^{-1} f^*(N) \tag{4}$$

where

$$R(N) = \frac{1}{N} \sum_{t=1}^{N} \varphi(t)\varphi(t)^T = \Phi\Phi^T \qquad f^*(N) = \frac{1}{N} \sum_{t=1}^{N} \phi(t) v(t)$$

Pro’s and Con’s of FIR Modeling

Pros.

LSE gives a consistent estimate $\lim_{N\to\infty} \hat{\theta}_N^{LS} = \theta_0$ as long as the input sequence $\{u(t)\}$ is uncorrelated with the noise term $v(t) = H(q)e(t)$, which may be colored and even correlated with $\{y(t)\}$.

Cons

The number of parameters, $m = n_b, \theta \in \Re^m$, may be too large to estimate. This may occur if

- The impulse response has a slow decaying mode, or
- The sampling rate is high.

The persistently exciting condition $rank\ \Phi = full\ rank$ can hardly be satisfied.

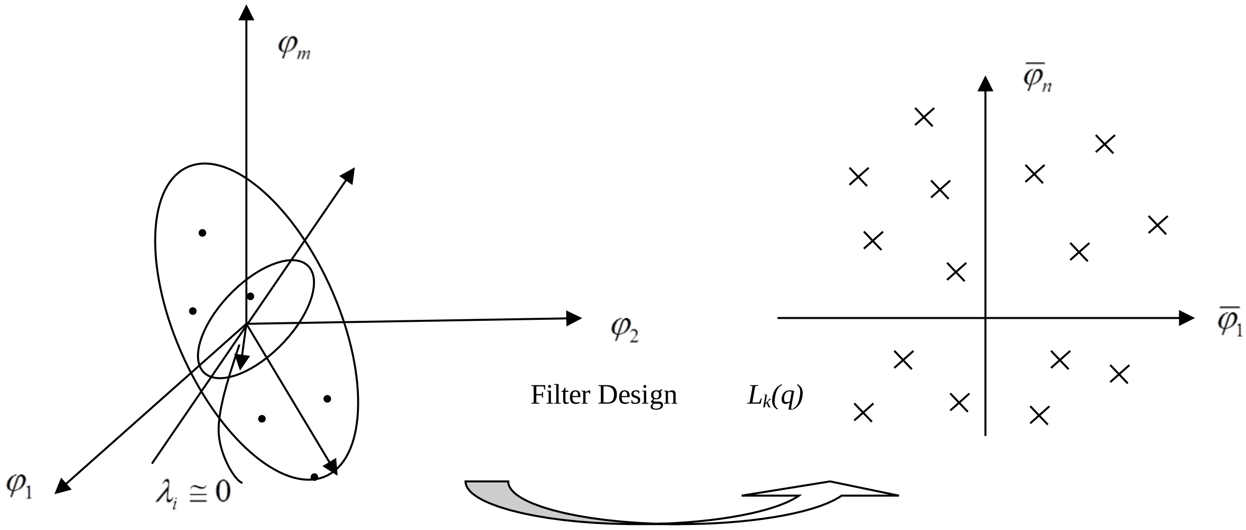
Check the eigenvalues of $\Phi\Phi^T$ (or the singular values of Φ)

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$$

It is likely $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_{n+1} \cong 0 = \cdots = \lambda_m$.

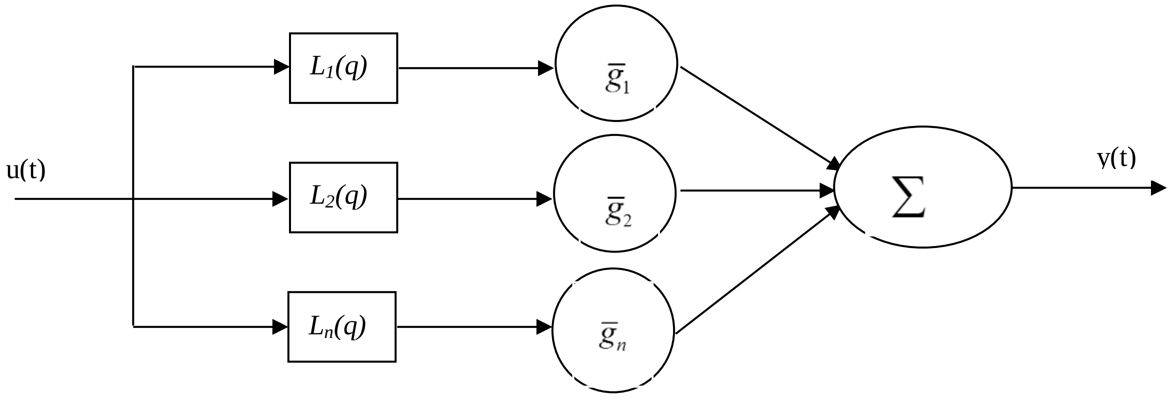
Often m becomes more than 50 and it is difficult to obtain such an input series having 50 non-zero singular values.

Time series data compression is an effective method for coping with this difficulty. Before formulating the above least square estimate problem, data are processed so that the information contained in the series of regressor may be represented in compact, low-dimensional form.



$R^m \longrightarrow R^n \quad n<m$

Coordinate Transformation (Data Compression)



15.2 Continuous-Time Laguerre Series Expansion

Let us begin with continuous-time Laguerre expansion.

[Theorem 15.1] If a transfer function $G(s)$ is

- Strictly proper $\lim_{s\to\infty} G(s) = 0 \quad G(\infty) = 0$ (5)
- Analytic in $\mathrm{Re}(s) > 0$ No pole on RHP
- Continuous in $\mathrm{Re}(s) \geq 0$

then, there exists a sequence $\{\bar{g}_k\}$ such that

$$G(s) = \sum_{k=1}^{\infty} \bar{g}_k \frac{\sqrt{2\bar{a}}}{s+\bar{a}} \left(\frac{s-\bar{a}}{s+\bar{a}} \right)^{k-1} \tag{6}$$

where $\bar{a} > 0$.

<u>Proof:</u>

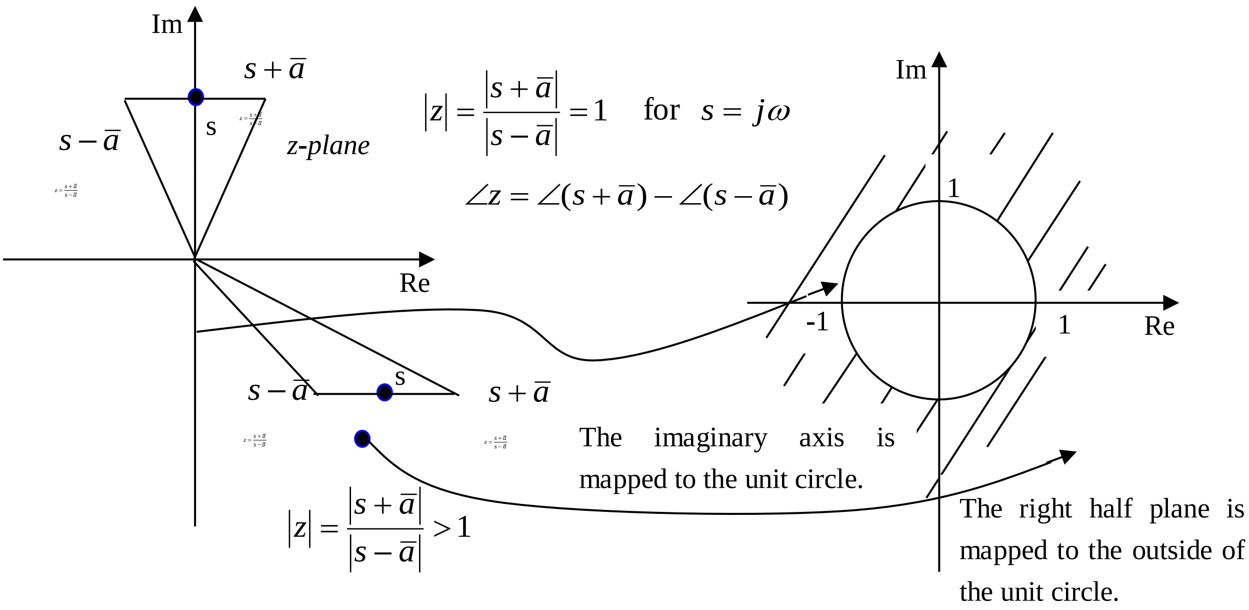
Consider the transformation given by

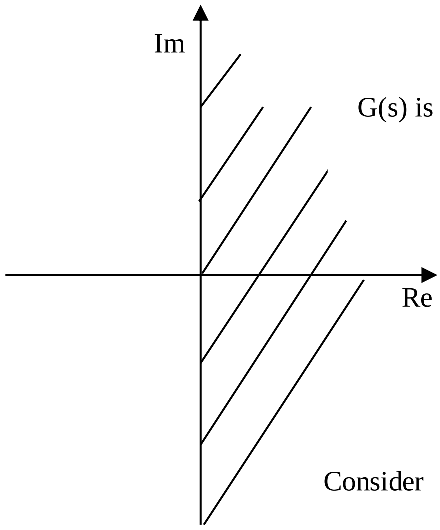
$$z = \frac{s+\bar{a}}{s-\bar{a}} \tag{7}$$

$$sz - \bar{a}z = s + \bar{a} \longrightarrow s(z-1) = \bar{a}(z+1)$$

The inverse transform is given by $\therefore s = \bar{a}\dfrac{z+1}{z-1}$ (8)

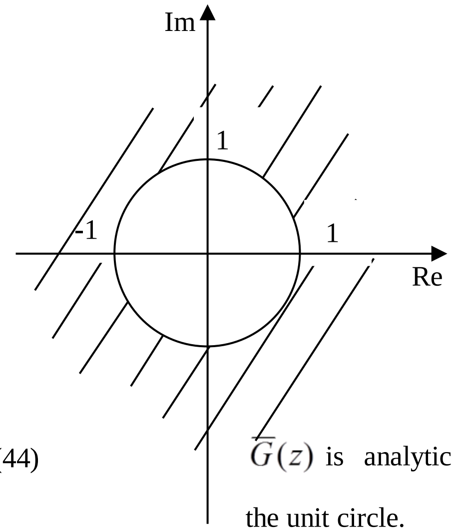
This called “bilinear transformation”.





G(s) is analytic in $\mathrm{Re}(s) > 0$

Consider $\quad \bar{G}(z) = G\left(\underbrace{\bar{a}\,\frac{z+1}{z-1}}_{s}\right) \quad$ (44)



$\bar{G}(z)$ is analytic outside the unit circle.

Therefore, there exists a Laurent expansion for $\bar{G}(z)$:

$$\bar{G}(z) = \sum_{k=1}^{\infty} g_k z^{-k} \qquad |z| > 1 \tag{9}$$

Example 1

$$G(s) = \frac{1}{(s-p_1)(s-p_2)}, \quad G\left(\bar{a}\,\frac{z+1}{z-1}\right) = \frac{1}{\left(\bar{a}\,\frac{z+1}{z-1} - p_1\right)\left(\bar{a}\,\frac{z+1}{z-1} - p_2\right)} \triangleq \bar{G}(z)$$

Considering that $\; G(\infty) = 0 \longrightarrow \; \lim_{s\to\infty} \frac{s+\bar{a}}{s-\bar{a}} = 1 \;\; \therefore \bar{G}(z) = 0$ at z = 1

This implies that $\bar{G}(z)$ has a factor of $(z-1)$ or $(1-z^{-1})z$

$$\bar{G}(z) = \frac{1}{\sqrt{2\bar{a}}}(1-z^{-1})\sum_{k=1}^{\infty} \bar{g}_k z^{-(k-1)} \tag{10}$$

Substituting (7) into (10)

$$\bar{G}\left(\frac{s+\bar{a}}{s-\bar{a}}\right) = G(s) = \frac{1}{\sqrt{2\bar{a}}}\left(1 - \frac{s-\bar{a}}{s+\bar{a}}\right)\sum_{k=1}^{\infty} \bar{g}_k \left(\frac{s+\bar{a}}{s-\bar{a}}\right)^{-(k-1)}$$

$$\therefore G(s) = \sum_{k=1}^{\infty} \bar{g}_k \frac{\sqrt{2\bar{a}}}{s+\bar{a}}\left(\frac{s-\bar{a}}{s+\bar{a}}\right)^{(k-1)}$$

$$L_k(s) = \frac{\sqrt{2\bar{a}}}{s+\bar{a}}\left(\frac{s-\bar{a}}{s+\bar{a}}\right)^{(k-1)} \tag{11}$$

Low Pass Filter $\qquad \left|\frac{s-\bar{a}}{s+\bar{a}}\right| \longrightarrow \left|\frac{j\omega-\bar{a}}{j\omega+\bar{a}}\right| = 1$ All-pass Filter

The Laplace transform of the Laguerre functions

The Main Point

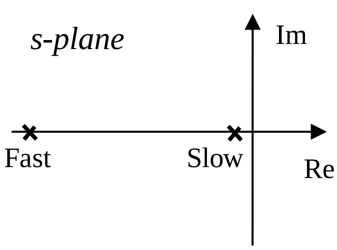
The above Laguerre series expansion can be used for data compression if the parameter $\bar{a}$, called a Laguerre pole, is chosen such that slow poles, i.e. dominant poles, of the original system are close to the Laguerre pole. Let p_i be a slow real pole of the original transfer function $G(s)$. If the Laguerre pole $\bar{a}$ is chosen such that $\bar{a} \cong |p_i|$, then a truncated Laguerre expansion:

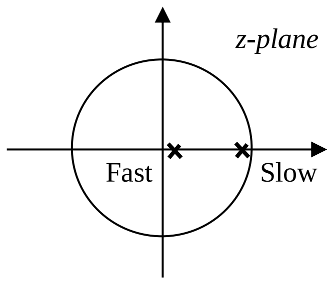
$$G_n(s) = \sum_{k=1}^{n} \bar{g}_k \frac{\sqrt{2\bar{a}}}{s+\bar{a}} \left(\frac{s-\bar{a}}{s+\bar{a}} \right)^{k-1} \rightarrow \quad G(s) \qquad (12)$$

converges to the original G(s) quickly for the following reason.

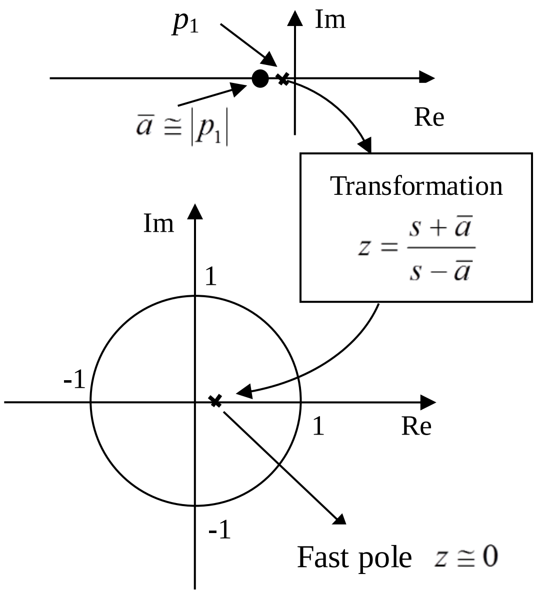
- Recall -

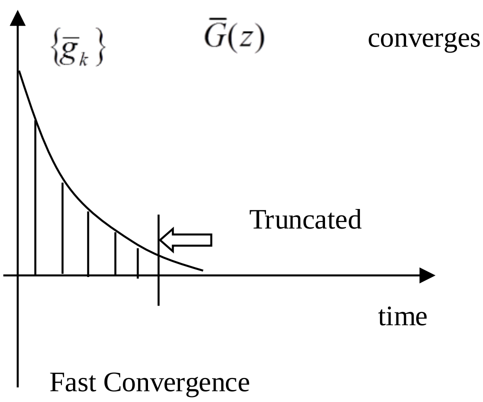
For a continuous-time system, a pole close to the imaginary axis is slow to converge, while a pole far from the imaginary axis converges quickly. Likewise, in discrete time, a pole close to the origin of a z-plane quickly converges, while the ones near the unit circle are slow.



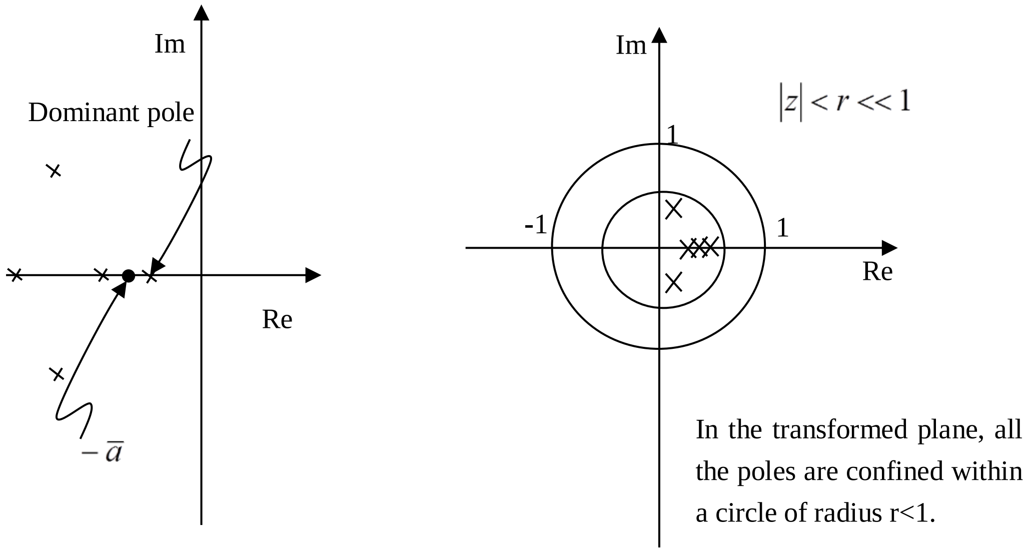


Choose $\bar{a}$ such that $p_1 \cong -\bar{a}$, where p_1 is a slow, stable pole. Using the bilinear transform, this slow pole in the s-plane can be transferred to a fast pole in the z-plane, as shown below. Representing in the z-plane, the transfer function can be truncated; just a few terms can approximate the impulse response since it converges quickly.





When the original transfer function has many poles, the Laguerre pole is placed near the dominant pole so that most of the slow poles may be approximated by the Laguerre pole.



With the bilinear transform, these slow poles are transferred to the ones near the origin of the z-plane. They are fast, hence the transfer function can be approximated to a low order model. The following example demonstrates these unique features of the Laguerre Series Expansion.

Example 2

Consider the continuous-time, Laguerre series expansion of the following two transfer functions:

$$G_1(s) = \frac{1}{(s+1)^2}, \quad G_2(s) = \frac{1}{(s+1)(s+2)}$$

a). The Laguerre series expansion is associated with the transformation of variables given by (7). Change the variable of the above transfer function $G_1(s)$ from *s* to *z*, and obtain the new transfer function $\overline{G}_1(z)$. Find the poles of $\overline{G}_1(z)$, when the parameter *a* is 1; $a = 1$.

Solution

Solving $z = \dfrac{s+a}{s-a}$ for *s*,

$$sz - az = s + a \qquad s(z-1) = a(z+1)$$

$$\therefore \quad s = a\frac{z+1}{z-1}$$

Replacing *s* by this in $G_1(s)$ yields:

$$\overline{G}_1(z) = G_1\left(a\frac{z+1}{z-1}\right) = \frac{1}{\left(a\frac{z+1}{z-1}+1\right)^2} = \frac{(z-1)^2}{\left(a(z+1)+(z-1)\right)^2}$$

$$\therefore \quad \overline{G}_1(z) = \frac{(z-1)^2}{\left((a+1)z+(a-1)\right)^2} \qquad \text{Poles} \quad z = -\frac{a-1}{a+1}, -\frac{a-1}{a+1}, \text{ Repeated poles}$$

b). Setting the parameter *a* to $a=1$, obtain the coefficients of the Laguerre series expansion of $G_1(s)$:

$$G_1(s) = \sum_{k=1}^{\infty} g_k \frac{\sqrt{2a}}{s+a}\left(\frac{s-a}{s+a}\right)^{k-1}$$

Is the coefficient series $\{g_k\}$ a finite series or an infinite series? What if the parameter *a* is not 1: $a \neq 1$? Explain why.

Solution

Substituting a = 1 into the above transfer function $\overline{G}_1(z)$ in part a) yields:

$$\overline{G}_1(z) = \frac{1}{4}(1-z^{-1})^2 = \frac{1}{4} - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}$$

This shows that the Laguerre series expansion converges in the second term. Therefore, it must be expressed as:

$$G_1(s) = \frac{1}{(s+1)^2} = g_1\frac{\sqrt{2}}{s+1} + g_2\frac{\sqrt{2}}{s+1}\frac{s-1}{s+1}$$
$$= \frac{\sqrt{2}(g_1+g_2)s + \sqrt{2}(g_1-g_2)}{(s+1)^2}$$

Multiplying $(s+1)^2$ to both sides, we obtain $1 = \sqrt{2}(g_1+g_2)s + \sqrt{2}(g_1-g_2)$.

Comparing the terms,

$$g_1 + g_2 = 0 \quad \therefore g_2 = -g_1$$

$$\sqrt{2}(g_1 - g_2) = 1 \therefore 2\sqrt{2}g_1 = 1 \Rightarrow g_1 = \frac{1}{2\sqrt{2}} = -g_2$$

$$\therefore g_1 = \frac{1}{2\sqrt{2}} \qquad g_2 = -\frac{1}{2\sqrt{2}}$$

c). Change the variable of $G_2(s)$ from s to z in the same way as part a), and obtain the new transfer function $\overline{G}_2(z)$. Find all the poles of the new transfer function $\overline{G}_2(z)$.

Solution

$$\overline{G}_2(z) = G_2\left(a\frac{z+1}{z-1}\right) = \frac{1}{\left(a\dfrac{z+1}{z-1}+1\right)\left(a\dfrac{z+1}{z-1}+2\right)}$$

$$= \frac{(z-1)^2}{(a(z+1)+(z-1))(a(z+1)+2(z-1))}$$

$$= \frac{(z-1)^2}{((a+1)z+(a-1))((a+2)z+(a-2))}$$

Poles $z_1 = -\dfrac{a-1}{a+1}, z_2 = -\dfrac{a-2}{a+2}$

d). Set $a = 1.5$, and plot the poles of $\overline{G}_2(z)$ on a complex plane. Next, set $a = 5$, and plot the poles once again. Which parameter value gives faster convergence in the Laguerre series expansion, $a = 1.5$ or $a = 5$? Explain mathematically why it converges more quickly than the other.

Solution

$a = 1.5$,

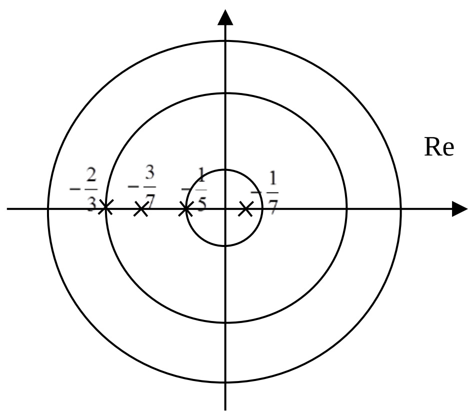
$$z_1 = -\frac{0.5}{2.5} = -\frac{1}{5},$$

$$z_2 = -\frac{-0.5}{3.5} = \frac{1}{7}$$

$a = 5$,

$$z_1 = -\frac{4}{6} = -\frac{2}{3},$$

$$z_2 = -\frac{3}{7}$$



The unit circle in the ***z*** - plane corresponds to the imaginary axis in the ***s*** - plane. If poles exist near the unit circle in the ***z*** - domain, convergence is slow. On the other hand, if poles are close to the origin in the ***z*** - domain, convergence is fast. Namely, the impulse response doesn't need many terms. An extreme case is that all the poles are at the origin of the ***z*** - domain. As shown in Part b), only two terms are needed when both poles are at the origin.

When $a = 1.5$, both poles are within a radius of $\frac{1}{5}$ from the origin in the complex plane, while those poles for $a = 5$ are outside of this radius. Therefore, the impulse response of $\overline{G_2}(z)$ with $a = 1.5$ converges more quickly than $\overline{G_2}(z)$ of $a = 5$.

15.3 Discrete-Time Laguerre Series Expansion

[Theorem 15.2]
Assume that a Z-transform G(z) is

- Strictly proper $G(\infty) = 0$
- Analytic in $|z| > 1$ RHP
- Continuous in $|z| \geq 1$

Then

$$G(z) = \sum_{k=1}^{\infty} \bar{g}_k \frac{K}{z-a}\left(\frac{1-az}{z-a}\right)^{k-1} \tag{13}$$

where -1 < a < 1 and

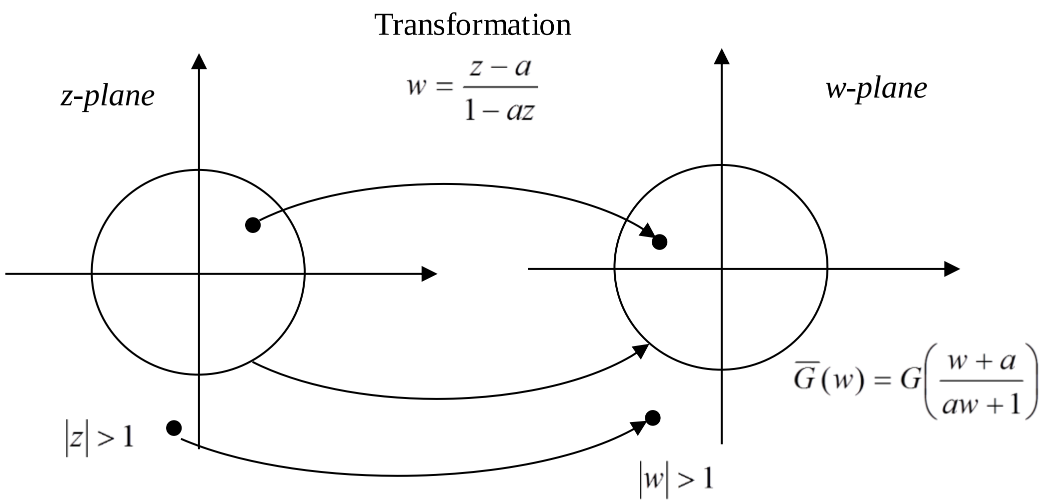
$$K = \sqrt{(1-a^2)T} \qquad T = \text{sampling Interval} \tag{14}$$

<u>Proof</u>
Consider the bilinear transformation:

$$w = \frac{z-a}{1-az} \tag{15}$$

$$w - azw = z - a \qquad w + a = z(aw+1) \ \text{ therefore, } \ z = \frac{w+a}{aw+1} \tag{16}$$

The relationship between *z*-plane and *w*-plane is more complex, but we can show the following correspondence between the two:



$\bar{G}(w) = G\left(\frac{w+a}{aw+1}\right)$ is analytic in $|w| > 1$, and is proper $G(\infty) = 0$

$$\lim_{z\to\infty} w = -\frac{1}{a} \qquad \overline{G}(-\frac{1}{a}) = 0 \tag{17}$$

Therefore

$$\overline{G}(w) = \frac{T}{K}(a + w^{-1})\sum_{k=1}^{\infty} \overline{g}_k w^{-(k-1)} \tag{18}$$

$$G(z) = \overline{G}\left(\frac{z-a}{1-az}\right) = \frac{T}{K}(a + \frac{1-az}{z-a})\sum_{k=1}^{\infty} \overline{g}_k \left(\frac{z-a}{1-az}\right)^{-(k-1)} \tag{19}$$

$$\therefore G(z) = \sum_{k=1}^{\infty} \overline{g}_k \frac{K}{z-a}\left(\frac{1-az}{z-a}\right)^{k-1}$$

Q.E.D.

Now we can write

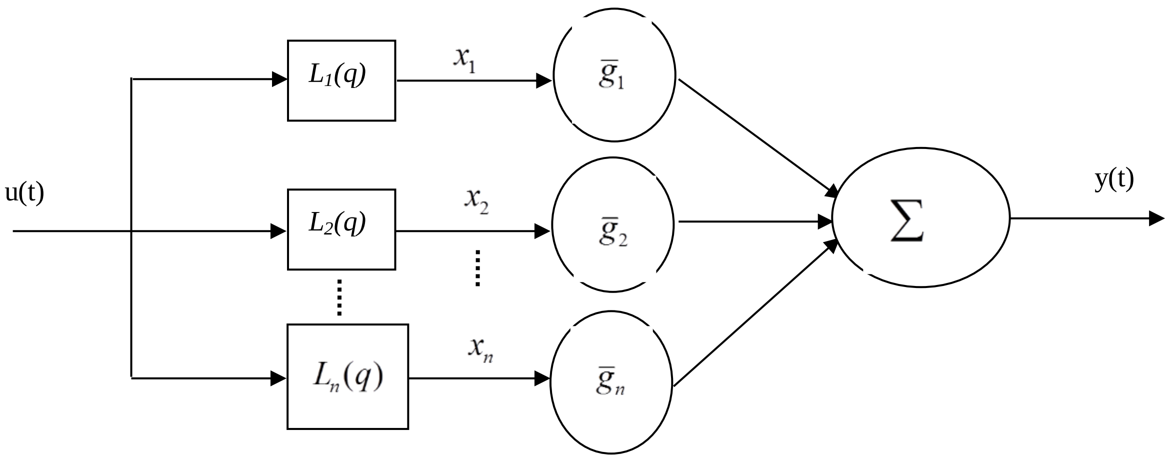
$$y(t) = G(q)u(t)$$

$$= \sum_{k=1}^{n} \overline{g}_k \frac{K}{q-a}\left(\frac{1-aq}{q-a}\right)^{k-1} u(t) = \sum_{k=1}^{n} \overline{g}_k L_k(q)u(t)$$

where $L_k(q), k = 1,\cdots,n$ is a series of filters. Once the original input data are filtered with $L_k(q), k = 1,\cdots,n$,

$$x_k = L_k(q)u(t), \quad k = 1,\cdots,n \tag{20}$$

The output $y(t)$ is represented as a moving average of the transformed input x_k, that is, a FIR model.



Furthermore, $x_k(t)$ can be computed recursively.

$$x_1(t) = L_1(q)u(t) = \frac{K}{q-a}u(t) = \frac{Kq^{-1}}{1-aq^{-1}}u(t)$$

$$x_1(t) - ax_1(t-1) = Ku(t-1)$$

$$x_1(t) = ax_1(t-1) + Ku(t-1)$$

$$x_2(t) = \frac{1-aq}{q-a}x_1(t) = \frac{q^{-1}-a}{1-aq^{-1}}x_1(t)$$

$$x_2(t) - ax_2(t-1) = x_1(t-1) - ax_1(t)$$

$$x_2(t) = ax_2(t-1) + x_1(t-1) - ax_1(t)$$

$$x_k(t) = ax_k(t-1) + x_{k-1}(t-1) - ax_{k-1}(t) \tag{21}$$

Recursive Filters