2.160 IDENTIFICATION, ESTIMATION, AND LEARNING

LECTURE NOTES NO. 10

PART 3: Linear System Identification

10. Introduction to System Identification

10.1 Overview

10.1.1Taxonomy We now move on to the second part of the course, System Identification. You will learn various techniques and theories for identifying systems. Figure 10-1 shows the classification and taxonomy of system identification methods, including both linear and nonlinear systems. Linear system identification methods are classified into parametric and non-parametric methods. The latter includes the method for determining either time response or frequency response directly from experimental data (direct method) and the one to take correlation between input and output signals (correlation method), which is robust against noise.

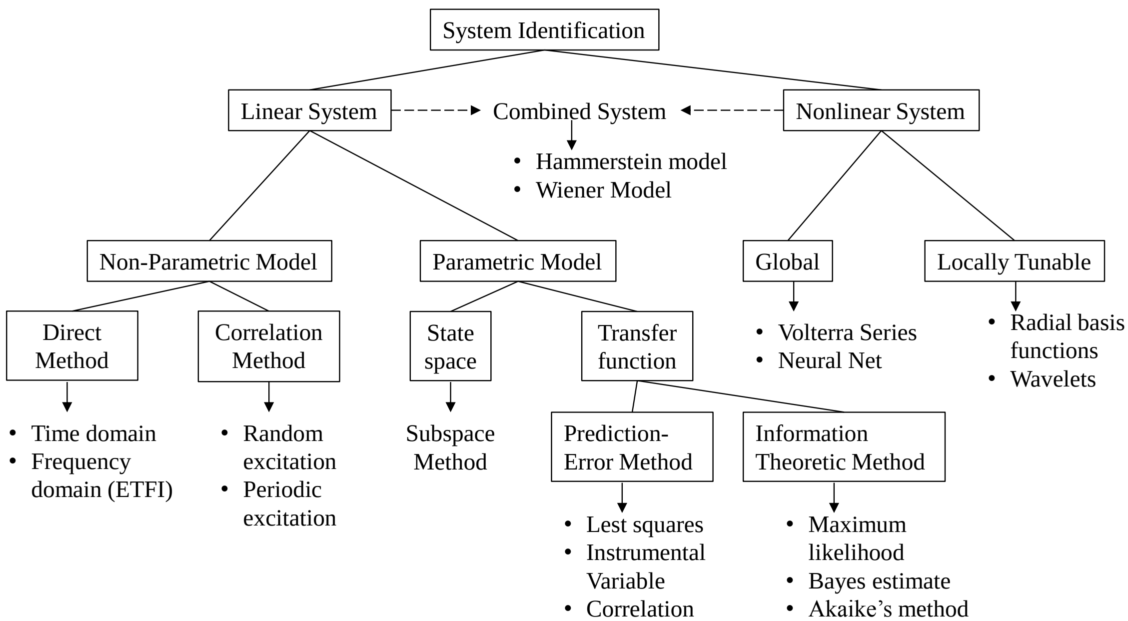


Figure 10-1 Taxonomy of system identification methods

Parametric models include both state space and transfer function models. The subspace method is a systematic approach to identifying state equations of a dynamical system. Transfer functions can be identified based on prediction error ($e(t) = \hat{y}(t) - y(t)$) or on the likelihood of a candidate model (Information theoretic method).

In general, nonlinear systems are represented by a collection of basis functions. Parameters associated with each basis function and the connectivity of the group of basis functions are identified to fit experiment data. Radial-basis functions and wavelets are well known as basis functions that can be tuned to local properties of a nonlinear system, while Volterra series expansion and artificial neural networks are tuned to the global input-output relationship of a nonlinear system. Hammerstein model and Wiener model represent a nonlinear system by a combination of linear dynamical system and nonlinear algebraic functions.

10.1.2 Issues, Main Results, and Theory Among many methods listed in Figure 10-1, we have to select a proper method to meet our goal. Fundamental questions include how we determine the right structure for representing a system; how we assure that data are good enough to identify the system; how we know the expected accuracy or variance of the identified model; and how we best design experiments.

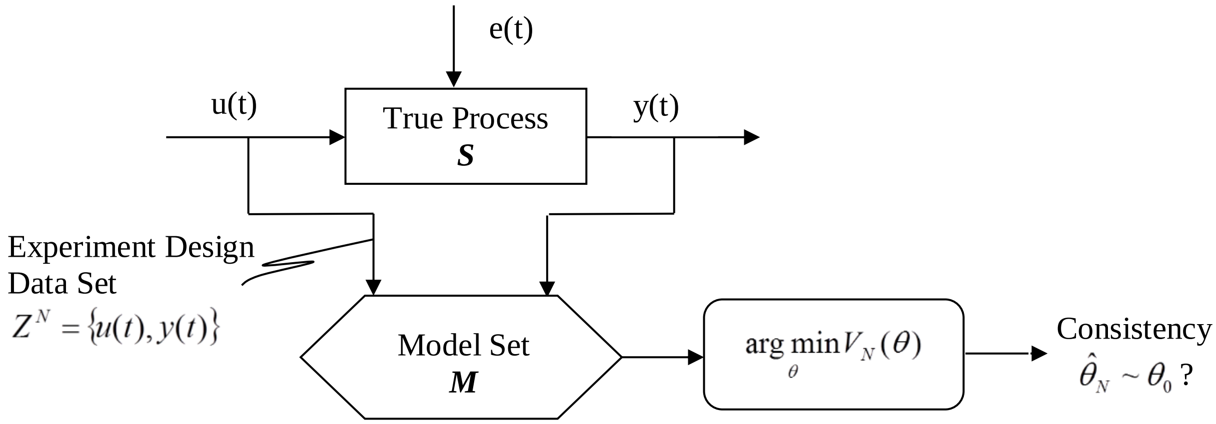


Figure 10-2 System identification process

Key Questions:

Q1: Is a given data set ***informative enough*** to uniquely determine a model from a given model set?

Does ***Z*** contain sufficient information to distinguish any two models in ***M***?

Q2: Is merely minimizing the mean squared prediction error $V_N(\theta)$ good enough to obtain the true (unbiased) model?

$$V_N(\theta) = \frac{1}{2N}\sum_{t=1}^{N}(y(t) - \hat{y}(t \mid \theta))^2$$

What if the true model is not involved in the model set?
How is the model-data fitting influenced by noise characteristics and input properties?

Q3: How accurate is the estimated model?
How much variance, expected error, etc.?
How much data needed?

Q4: How can we design experiments?

Key Results of system identification theory

- Asymptotic variance and confidence intervals
- Model structure and model-order reduction

- Nonlinear function approximation: Radial basis functions, neural networks
- Frequency-domain analysis: Informative experiment and persistent excitation
- Input design: Pseudo Random Binary signal
- Maximum likelihood and information theoretic approach
- Accuracy-variance trade-off: Akaike's Information Criterion

Mathematical tools for Part 2 system identification

- Discrete Fourier transform and spectral analysis
- Central limit theorems
- Random processes: wide-sense stationary process, ergodic process, etc.

Identification and estimation are closely related. The same Prediction-Error approach used for estimation will be applied to identification. A few mathematical tools must be reviewed before discussing identification theory and techniques, however.

10.2 Review of Mathematical Tools

10.2.1 Impulse Response and Transfer Operator

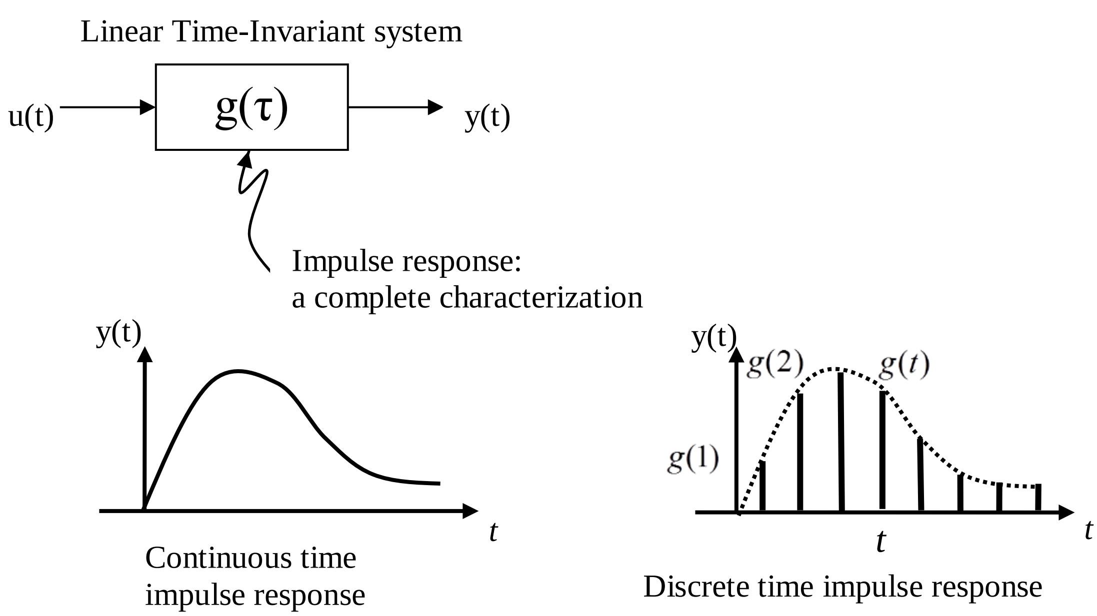


Figure 10-3 Impulse response

Any linear time-invariant system can be completely characterized with Impulse Response: $g(t)$. For an arbitrary input, $\{u(s)|s \le t\}$, the output $y(t)$ is given by the convolution of the input and the impulse response given by

$$\text{Continuous Time Convolution} \quad y(t) = \int_0^\infty g(\tau)u(t-\tau)d\tau \tag{1}$$

For discrete-time systems, an impulse response is given by an infinite time series:

$g(0), g(1), g(2), \cdots$, or $\{g(\tau) | \tau = 0 \ldots \infty\}$

$\{u(s) | s \le t\}$ $\Rightarrow$ Output $\{y(s) | s \le t\}$

Discrete Time Convolution

$$y(t) = \sum_{k=0}^{\infty} g(k) u(t-k) \tag{2}$$

> Continuous Time Differential Operator
>
> $p = \dfrac{d}{dt}$
>
> $\ddot{y} + 2\dot{y} + 3y = \dot{u} + 2u$
>
> $(p^2 + 2p + 3) y(t) = (p+2) u(t)$
>
> $y(t) = \dfrac{p+2}{p^2 + 2p + 3} u(t) = G(p) u(t)$

Similar to the continuous-time differential operator shown above right, which leads to transfer function *G*(*p*), use of the following discrete-time shift operator, *q*, provides a discrete-time transfer function.

Forward shift operator: *q:* $qu(t) = u(t+1)$ (3)

Backward shift operator: q^{-1} : $q^{-1} u(t) = u(t-1)$ (4)

Using (3) and (4) in (2)

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} g(k) \left(q^{-k} u(t) \right) \\ &= \underbrace{\sum_{k=0}^{\infty} \left(g(k) q^{-k} \right)}_{G(q)} u(t) = G(q) u(t) \end{aligned} \tag{5}$$

Transfer Operator or Transfer Function

Monic Transfer Function: $G(q) = 1 + \sum_{k=1}^{\infty} g(k) q^{-k} = \sum_{k=0}^{\infty} g(k) q^{-k} \qquad w / g(0) = 1$

10.2.2 *Z*-Transform

Taking Laplace transform of (2) yields

$$L[y] = \sum g(k) \underbrace{e^{-k\Delta T s}}_{(e^{\Delta T s})^{-k}} L[u]$$

where ΔT is a sampling interval. Replacing $e^{\Delta T s}$ by z yields the z-transform of the transfer function

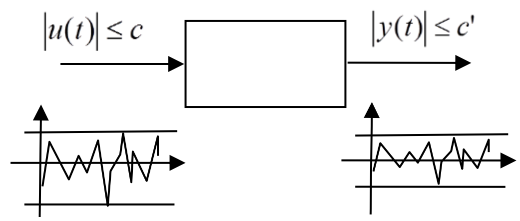
$$G(z) = \sum_{k=0}^{\infty} g(k) z^{-k} \tag{6}$$

which is a complex function of $z = e^{\Delta T s}$. Poles and zeros are defined as:

A zero of $G(z)$ is a complex number z_i that makes $G(z)$ zero: $G(z_i)=0$
A pole of $G(z)$ is a complex number z_j that makes $G(z)$ infinite $G(z_j)=\infty$

Bounded-input, bounded output (BIBO) stability

Figure 10-4 BIBO stability



[Theorem] A linear time-invariant system with transfer function $G(q)=\sum_{k=1}^{\infty} g(k)q^{-k}$
is BIBO stable if

$$\sum_{k=1}^{\infty} |g(k)| < \infty \qquad (7)$$

<u>Proof</u>: Suppose that $|u(t)| \le c$, then

$$|y(t)| = \left|\sum_{k=1}^{\infty} g(k)u(t-k)\right| \le \sum_{k=1}^{\infty} |g(k)u(t-k)| \le c\sum_{k=1}^{\infty} |g(k)| \le c'$$

The output is bounded; the condition of (7) satisfies the BIBO stability criterion. Q.E.D.

Input $|u(t)| \le c$ (Bounded) $\longrightarrow$ Output $|y(t)| \le c'$ (Bounded)

$$\uparrow$$

$$\sum_{k=1}^{\infty} |g(k)| < \infty$$

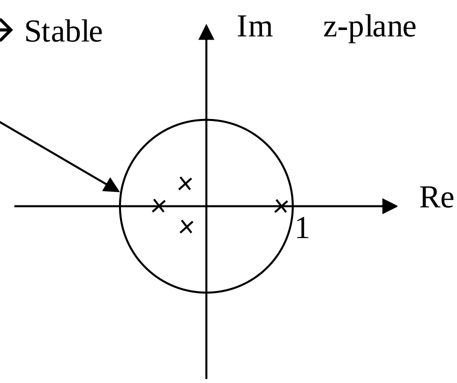
In the z-domain, this BIBO stability is interpreted as follows. Associated with $G(q)$, consider

$$G(z)=\sum_{k=0}^{\infty} g(k)z^{-k}$$

Under the condition of (7), this $G(z)$ converges for all $|z| \ge 1$

$$\Downarrow$$

$G(z)$ has <u>no poles</u> on and outside <u>the unit circle</u>. → Stable



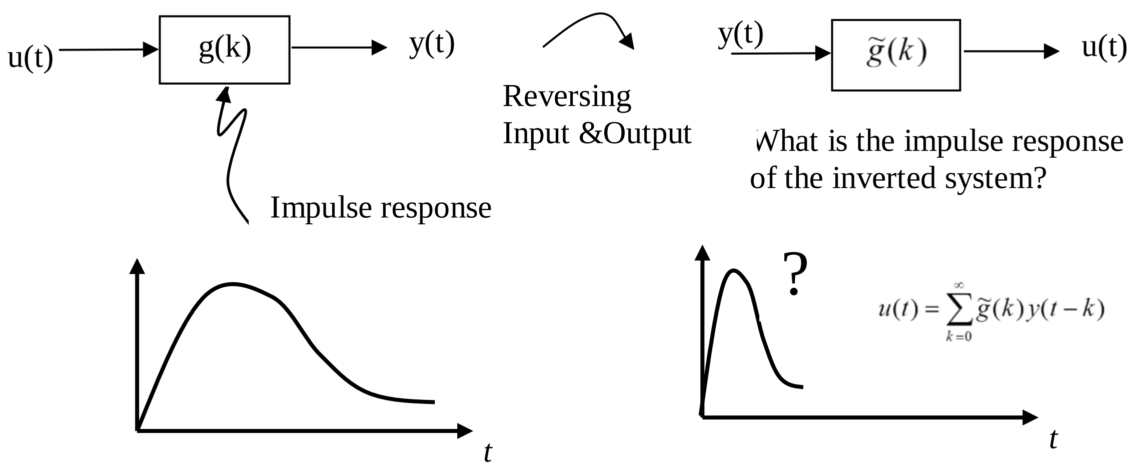
If poles exist, they must be within the unit circle.

This stability criterion can be shown easily. For $|z| > 1$,

$$|G(z)| \le \sum_{k=1}^{\infty} \left| g(k) z^{-k} \right| \le \sum_{k=1}^{\infty} \frac{|g(k)|}{|z|^k} < \sum_{k=1}^{\infty} |g(k)| < \infty$$

Therefore, there is no pole on and outside the unit circle, if the system is BIBO stable. Consider (6) as a Laurent expansion, the above result means that function $G(z)$ is analytic on and outside the unit circle.

Inverse Transfer Function



Consider the z-transform of stable $G(q)$: $G(z) = \sum_{k=0}^{\infty} g(k) z^{-k}$. Assume that function $\frac{1}{G(z)}$ is analytic in $|z| \ge 1$, then it can be expressed in the Laurent expansion form

$$\frac{1}{G(z)} = \sum_{k=0}^{\infty} \tilde{g}(k) z^{-k} \tag{8}$$

These $\tilde{g}(k) \quad k = 0 \ldots \infty$ give the impulse response of the inverse system. We write

$$G^{-1}(q) = \sum_{k=0}^{\infty} \tilde{g}(k) q^{-k} \tag{9}$$

Then $G(q)$ is said to be inversely stable.

Example:

Obtain the inverse transfer function of the following process:

$$y(t) = u(t) + cu(t-1), \quad |c| < 1$$

Solution: $y(t) = u(t) + cq^{-1}u(t) = (1 + cq^{-1})u(t) \Rightarrow \quad G(q) = 1 + cq^{-1}$

z-transform $G(z) = 1 + cz^{-1} = \frac{z+c}{z}$

Inverting G(z)

$$G^{-1}(z) = \frac{1}{G(z)} = \frac{z}{z+c} = \frac{1}{1+cz^{-1}} = \sum_{k=0}^{\infty} (-c)^k z^{-k}$$

Therefore $u(t) = \sum_{k=0}^{\infty} (-c)^k y(t-k)$, $\tilde{g}(k) = (-c)^k$.

11. Non-Parametric Linear System identification

We begin with non-parametric system identification of linear time-invariant systems. One of the challenges in system identification is to select the right model structure. State equations and transfer functions use a particular parametric structure for representing the system effectively. However, it is often difficult to determine the order of the system, the number of poles and zeros, or the group of parameters that represent the system properly. On the other hand, we know that a linear time-invariant system can be represented by an impulse response, a Bode diagram, or other means, which do not require a selection of model structure. System identification without an explicit selection of parametric model structure is referred to as Non-Parametric Identification.

11.1 Direct Methods

11.1.1 Impulse Response Method Consider a linear time-invariant system :

$$y(t) = \sum_{k=0}^{\infty} g(k)u(t-k) \tag{1}$$

Suppose that an impulse is given to the system:

$$u(t) = \begin{cases} c: & t = 0 \\ 0: & t \neq 0 \end{cases} \tag{2}$$

Then the output will be

$$y(t) = cg(t) + v(t) \tag{3}$$

where the second term $v(t)$ is noise observed at the output. The impulse response $\{g(t), t \geq 0\}$ can be determined as

$$\hat{g}(t) = \frac{1}{c} y(t) \tag{4}$$

However, the noise term remains in the estimate of impulse response values:

$$g(t) = \frac{1}{c} y(t) - \frac{1}{c} v(t) \tag{5}$$

We should use the maximum allowable input amplitude c for reducing the effect of noise.

10.1.2 Sine-Wave Response The frequency transfer function of $G(s)$ is given by replacing s by $j\omega : G(j\omega)$. In discrete time, z-transform transfer function $G(z)$ can be converted to the corresponding frequency transfer function $G(e^{j\omega}); -\pi \leq \omega < \pi$ by replacing z by $e^{j\omega}$.

$$G(z) = \sum_{k=0}^{\infty} g(k) z^{-k} \quad \Rightarrow \quad G(e^{j\omega}) = \sum_{k=0}^{\infty} g(k) e^{-j\omega k} \tag{6}$$

Consider sine waves for determining the frequency transfer function in discrete time:

$$u(t) = A\sin\omega t = A\,\mathrm{Im}\,e^{j\omega t} \tag{7}$$

Substituting this into (10-2) yields

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} g(k)u(t-k) = A\sum_{k=0}^{\infty} g(k)\,\mathrm{Im}\,e^{j\omega(t-k)} = A\,\mathrm{Im}\left[e^{j\omega t}\sum_{k=0}^{\infty} g(k)e^{-j\omega k}\right] \\ &= A\,\mathrm{Im}\left[e^{j\omega t}G(e^{j\omega})\right] = A\left|G(e^{j\omega})\right|\sin(\omega t + \phi) \end{aligned} \tag{8}$$

Measuring the amplitude and phase angle of output $y(t)$ for each of many frequencies, we can determine the frequency transfer function. Note, however, that the output noise term $v(t)$ remains as in the case of (5), which causes error in determining the transfer function.

11.2 Correlation Method

Correlation Method allows us to eliminate the effect of noise. In this method, an input time sequence that is uncorrelated with noise is given to the system, and the correlation between input $u(t)$ and output $y(t)$ is computed. The idea is that the noise component involved in the output is filtered out, as it is uncorrelated with the input.

Recall the auto-correlation of a wide-sense stationary process:

$$R_u(\tau) = E\left[u(t)u(t+\tau)\right] \tag{9}$$

If the process is ergodic, the ensemble mean $E[\bullet]$ can be replaced by time average:

$$R_u(\tau) = \lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T} u(t)u(t+\tau)dt\right] \tag{10}$$

or in discrete time

$$R_u(\tau) = \lim_{N\to\infty}\left[\frac{1}{2N+1}\sum_{t=-N}^{N} u(t)u(t+\tau)\right] \tag{11}$$

The cross-correlation between the input and output is given by

$$R_{uy}(\tau) = E\left[u(t)y(t+\tau)\right] = \lim_{N\to\infty}\left[\frac{1}{2N+1}\sum_{t=-N}^{N} u(t)y(t+\tau)\right] \tag{12}$$

Using (10-2) we can rewrite the output term as

$$y(t+\tau) = \sum_{k=0}^{\infty} g(k)u(t+\tau-k) \tag{13}$$

Therefore, the cross-correlation is given by

$$R_{uy}(\tau) = \lim_{N\to\infty}\left[\frac{1}{2N+1}\sum_{t=-N}^{N} u(t)\sum_{k=0}^{\infty} g(k)u(t+\tau-k)\right]$$
$$= \sum_{k=0}^{\infty} g(k)\underbrace{\lim_{N\to\infty}\left[\frac{1}{2N+1}\sum_{t=-N}^{N} u(t)u(t+\tau-k)\right]}_{R_u(\tau-k)} \tag{14}$$

$$\therefore \quad R_{uy}(\tau) = \sum_{k=0}^{\infty} g(k)R_u(\tau-k) \tag{15}$$

This is called the Wiener-Hopf Equation:

If the system is BIBO stable: $\lim_{k\to\infty} g(k) = 0$. Truncating $g(k)$ at $k = N$ yields

$$\begin{aligned}
R_{uy}(0) &= g(0)R_u(0) + g(1)R_u(-1) + \cdots + g(N)R_u(-N) \\
R_{uy}(1) &= g(0)R_u(1) + g(1)R_u(0) + \cdots + g(N)R_u(-N+1) \\
&\vdots \\
R_{uy}(N) &= g(0)R_u(N) + g(1)R_u(N-1) + \cdots + g(N)R_u(0)
\end{aligned} \tag{16}$$

which can be arranged in vector-matrix form

$$\begin{bmatrix} R_{uy}(0) \\ R_{uy}(1) \\ \vdots \\ R_{uy}(N) \end{bmatrix} = \underbrace{\begin{bmatrix} R_u(0) & R_u(-1) & \cdots & R_u(-N) \\ R_u(1) & R_u(0) & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ R_u(N) & \cdots & \cdots & R_u(0) \end{bmatrix}}_{\mathbf{R}_N} \cdot \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N) \end{bmatrix} \tag{17}$$

If the matrix is of full rank, (17) can be solved for the coefficients of Finite Impulse Response:

$$\begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N) \end{bmatrix} = \mathbf{R}_N^{-1} \cdot \begin{bmatrix} R_{uy}(0) \\ R_{uy}(1) \\ \vdots \\ R_{uy}(N) \end{bmatrix} \tag{18}$$

Various signals can be used for the input time sequence, which must be uncorrelated with noise.

11.2.1 Random Signal Consider to use a random signal with variance λ:

$$\left.\begin{array}{l} R_u(0) = \lambda \\ R_u(1) = 0 \\ \quad\vdots \\ R_u(N) = 0 \end{array}\right\} \tag{19}$$

Note that this input signal is uncorrelated with time. Substituting this into (17) yields

$$g(t) = \frac{1}{\lambda} R_{uy}(t), \quad 0 \le t \le N \tag{20}$$

Now let us examine how the noise term is eliminated in this method. To this end we write the output as the sum of noise-free component $\bar{y}(t)$ and noise:

$$y(t) = \bar{y}(t) + v(t) \tag{21}$$

Substituting this into the cross-correlation (12)

$$\begin{aligned} R_{uy}(\tau) &= \frac{1}{2N+1} \sum_{k=-N}^{N} u(k) y(\tau + k) \\ &= \frac{1}{2N+1} \sum_{k=-N}^{N} u(k)(\bar{y}(k+\tau) + v(k+\tau)) \\ &= R_{u\bar{y}}(\tau) + \frac{1}{2N+1} \sum_{k=-N}^{N} u(k) v(k+\tau) \end{aligned} \tag{22}$$

It is clear that the second term, the noise term, vanishes as it is uncorrelated with the input sequence.

$$\frac{1}{2N+1} \sum_{k=-N}^{N} u(k) v(k+\tau) = 0 \tag{23}$$

Therefore, the Wiener-Hopf relationship can filter out the noise term uncorrelated with the input effectively.

11.2.2 Sine-Cosine Signals Periodic signals, such as sine and cosine waves, may be uncorrelated with noise and are thereby usable for the noise-free identification. Here we use sine and cosine waves as inputs and take correlation with the individual outputs.

$$I_S(N) = \frac{1}{NT} \sum_{t=0}^{NT} y(t) \sin \omega t, \quad I_C(N) = \frac{1}{NT} \sum_{t=0}^{NT} y(t) \cos \omega t \tag{24}$$

Using (21) and (8),

$$
\begin{aligned}
I_S(N) &= \frac{1}{NT}\sum_{t=0}^{NT} A\left|G(e^{j\omega})\right| \underbrace{\sin(\omega t+\phi)\sin\omega t}_{\frac{1}{2}\left(\cos\phi-\cos(2\omega t+\phi)\right)} + \frac{1}{NT}\sum_{t=0}^{NT} v(t)\sin\omega t \\
&= \frac{A}{2}\left|G(e^{j\omega})\right|\cos\phi - \underbrace{\frac{A}{2}\left|G(e^{j\omega})\right|\frac{1}{NT}\sum_{t=0}^{NT}\cos(2\omega t+\phi)}_{0,\text{ as } N\to\infty} + \underbrace{\frac{1}{NT}\sum_{t=0}^{NT} v(t)\sin\omega t}_{0,\text{ as } N\to\infty}
\end{aligned}
\tag{25}
$$

Note that the second term vanishes for sufficiently long time-average. The noise term, too, vanishes, as it is uncorrelated with the sine wave. A similar result can be obtained for the cosine wave.

Combining the above results for $I_S(N)$ and $I_C(N)$ we can obtain the gain and phase of the frequency transfer function without noise:

$$
\left|\hat{G}(e^{j\omega})\right| = \frac{2}{A}\sqrt{I_S^2+I_C^2}, \quad \hat{\phi} = \arg\hat{G}(e^{j\omega}) = \tan^{-1}\frac{I_C(N)}{I_S(N)}
\tag{26}
$$

where

$$
I_S(N) = \frac{2}{A}\left|\hat{G}(e^{j\omega})\right|\cos\phi, \quad I_C(N) = \frac{2}{A}\left|\hat{G}(e^{j\omega})\right|\sin\phi, \quad I_S^2+I_C^2 = \frac{4}{A^2}\left|\hat{G}(e^{j\omega})\right|^2
\tag{27}
$$

The above procedure must be repeated for different frequencies. Although the gain and phase of each frequency is noise-free in theory, the resultant Bode plots tend to be not smooth. Effective techniques, such as the Hamming Window, have been used widely to obtain smooth Bode plots.