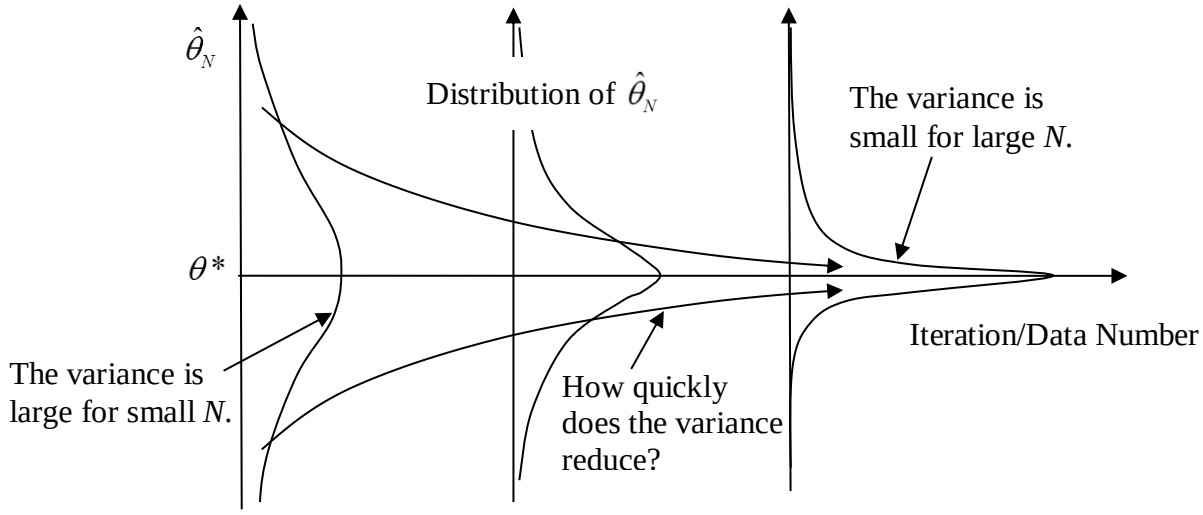
2.160 IDENTIFICATION, ESTIMATION, AND LEARNING

LECTURE NOTES NO. 13

14 Asymptotic Distribution of Parameter Estimates

14.1 Overview

Now that the identification problem of parametric models has been formulated based on the Prediction-Error Method (PEM), this chapter will analyze basic properties of the parameter estimation for a few simple model structures. Among others fundamental questions about the PEM parameter estimation include whether the estimation process converges; if converges, how quickly it converges; how accurately it can be estimated; what will be the expected variance; and how many data must be taken to guarantee certain accuracy. See the figure below. Asymptotic Variance Analysis gives you a guideline for addressing these issues.



The main points to be obtained in this chapter

The variance analysis of this chapter will reveal

a) The estimate converges to θ^* at a rate proportional to $\frac{1}{\sqrt{N}}$
b) Distribution converges to a Gaussian distribution: ***N***(0, Q).
c) Confidence Interval and Confidence Level
d) Cov $\hat{\theta}_N$ depends on a signal-to-noise ratio.

Identified model parameter $\hat{\theta}_N$ with cov $\hat{\theta}_N$: a "quality tag" confidence interval

14.2 Central Limit Theorems.

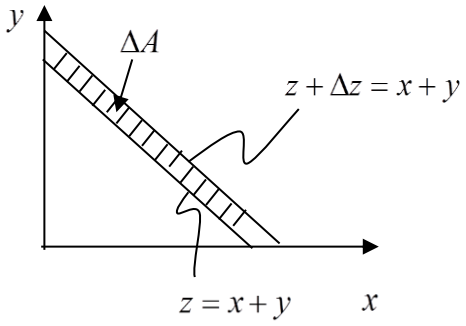
The mathematical tool needed for asymptotic variance analysis is "Central Limit" theorems. The following is a quick review of the theory.

Consider two independent random variable, *X* and *Y*, with PDF, $f_X(x)$ and $f_Y(y)$. Define another random variable *Z* as the sum of *X* and *Y*:

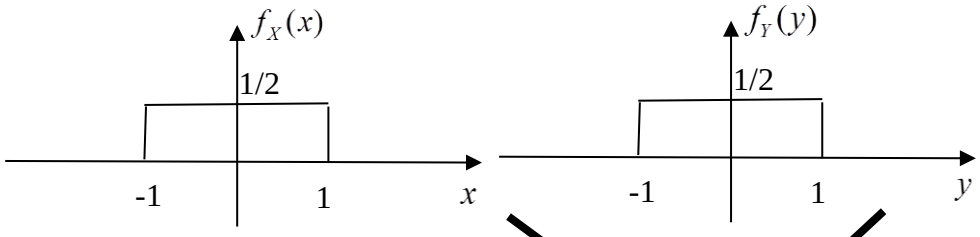
$$Z = X + Y$$

Let us obtain the PDF of *Z*.

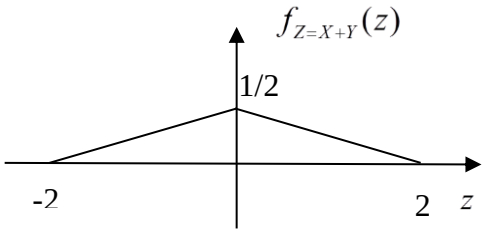
$$\begin{aligned}
&\Pr\left(z \le Z \le z + \Delta z\right) \\
&\quad = \iint_{\Delta A} f_X(x) f_Y(y) dx dy \\
&\quad = \left[\int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx\right] \Delta z = f_Z(z) \Delta z
\end{aligned}$$



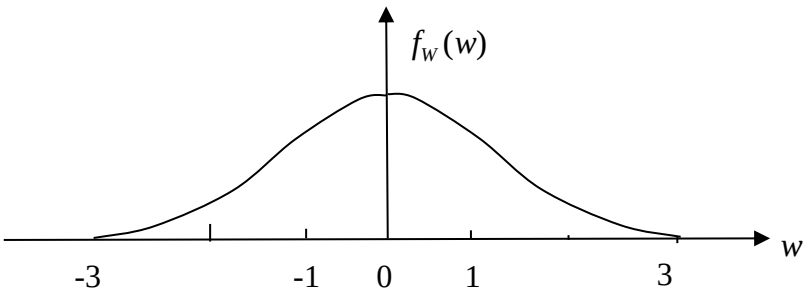
Example



$f_X(x)$ and $f_Y(y)$ have the same uniform distribution. Combining the two distributions, we can obtain the distribution of *Z*.



Further, consider $W = X + Y + V$, $f_V(v)$ has the same rectangular PDF as *X* and *Y*.



The resultant PDF is getting close to a Gaussian distribution.

In general, the PDF of a random variable $\sum_{i=1}^{N} X_i$ approaches a Gaussian distribution, regardless of the PDF of each X_i, as *N* gets larger. More rigorously, the following central limit theorem has been proven.

A Central Limit Theorem of Independent Random Variables

Let $X_t, t = 1, 2, \cdots$ be *d*-dimensional random variables with

$$\text{Mean} \qquad m = E(X_t)$$

$$\text{Co-variance} \quad C_X = E\left[\left(X_t - m\right)\left(X_t - m\right)^T\right] \quad \text{for all } t \tag{1}$$

Consider the sum of $X_t - m$ given by

$$Y_N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \left(X_t - m\right) \tag{2}$$

Then, as *N* tends to infinity, the distribution of Y_N converges to the Gaussian distribution given by PDF:

$$f_Y(y) = \frac{1}{(2\pi)^{d/2}\sqrt{\det C_X}} \exp\left\{-\frac{1}{2} y^T C_X{}^{-1} y\right\} \tag{3}$$

where

$$y = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \left(X_t - m\right).$$

14.3 Distribution of Estimate $\hat{\theta}_N$

Applying the Central Limit Theorem, we can obtain the distribution of estimate $\hat{\theta}_N$ as *N* tends to infinity.

Let $\hat{\theta}_N$ be an estimate based on the prediction error method (PEM);

$$\hat{\theta}_N = \arg\min_{\theta \in D_M} V_N(\theta, Z^N) \tag{4}$$

$$V_N(\theta, Z^N) = \frac{1}{N} \sum_{t=1}^{N} \frac{1}{2} \varepsilon^2(t, \theta) \quad \text{, where } \varepsilon(t, \theta) = y(t) - \hat{y}(t \mid \theta) \tag{5}$$

For simplicity, we first assume that the predictor $\hat{y}(t|\theta)$ is given by a linear regression:

$$\hat{y}(t|\theta) = \varphi^T \theta \tag{6}$$

and the parameter vector of the true system, θ_0, is involved in the model set, $\theta_0 \in D_M$.

The actual data is generated by

$$y(t) = \phi^T \theta_0 + e_0(t) \tag{7}$$

where

$$E[e_0(t)e_0(s)] = \begin{cases} \lambda_0 & t = s \\ 0 & t \neq s \end{cases}.$$

Since $\hat{\theta}_N$ minimizes $V_N(\theta, Z^N)$

$$V'_N(\hat{\theta}_N, Z^N) = \frac{d}{d\theta} V_N(\theta_N, Z^N)\Big|_{\theta=\hat{\theta}_N} = 0, \qquad V'_N \in R^{d\times 1} \tag{8}$$

Using the Mean Value Theorem, V'_N can be expressed as

$$V'_N(\hat{\theta}_N, Z^N) = V'_N(\theta_0, Z^N) + V''_N(\xi_N, Z^N)(\hat{\theta}_N - \theta_0) \qquad \hat{\theta}_N \le \xi_N \le \theta_0 \; or \quad \theta_0 \le \xi_N \le \hat{\theta}_N \tag{9}$$

where ξ_N is a parameter vector somewhere between θ_0 and $\hat{\theta}_N$.

Assuming that $V''_N(\xi_N, Z^N) = \frac{d}{d\theta} V'_N$ is non-singular and using (8) for (9),

$$\hat{\theta}_N - \theta_0 = -\left[V''_N(\xi_N, Z^N)\right]^{-1} V'_N(\theta_0, Z^N) \tag{10}$$

To obtain the distribution of $\hat{\theta}_N - \theta_0$, let us first examine $V'_N(\theta_0, Z^N)$ as N tends to infinity.

$$V'_N(\theta_0, Z^N) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon(t, \theta_0) \frac{d\varepsilon}{d\theta}\Big|_{\theta=\theta_0}, \tag{11}$$

Using (6) and recalling $\varepsilon(t, \theta) = y(t) - \hat{y}(t \mid \theta)$

$$\frac{d\varepsilon}{d\theta}\Big|_{\theta_0} = -\frac{d}{d\theta}\hat{y}(t|\theta)\Big|_{\theta_0} = -\varphi^T(t), \tag{12}$$

and

$$\varepsilon(t, \theta_0) = \varphi^T(t)\theta_0 + e_0(t) - \varphi^T(t)\theta_0 = e_0(t)$$

Therefore, (11) reduces to

$$-V'_N(\theta_0, Z^N) = \frac{1}{N}\sum_{t=1}^{N}\varphi(t)e_0(t) \tag{13}$$

Let us treat $\varphi(t)e_0(t) \equiv X_t$ as a random variable. Its mean is zero, since

$$m = \bar{E}\left[\varphi(t)e_0(t)\right] = \bar{E}\left[\varphi(t)\right]\bar{E}\left[e_0(t)\right] = 0 \tag{14}$$

The covariance is

$$\begin{aligned} C_X(t,s) &= \bar{E}\left[\left(X_t - m\right)\left(X_s - m\right)^T\right] = \bar{E}\left[\varphi(t)e_0(t)e_0(s)\varphi(s)^T\right] \\ &= \begin{cases} E[e_0(t)]\bar{E}\left[\varphi(s)e_0(s)\varphi(s)^T\right] = 0, \text{ for } t > s \\ E[e_0(s)]\bar{E}\left[\varphi(t)e_0(s)\varphi(s)^T\right] = 0, \text{ for } s > t \end{cases} \\ &= 0, \quad \text{for } t \neq s \end{aligned} \tag{15}$$

$$C_X(t,t) = \bar{E}\left[e_0^2(t)\right]\bar{E}\left[\phi(t)\phi(t)^T\right] = \lambda_0\bar{R} \tag{16}$$

Note that $X_1, X_2, \ldots, X_N$ are independent, since $e_0(t)$ is independent.

Consider

$$Y_N = \frac{1}{\sqrt{N}}\sum_{t=1}^{N}\left(X_t - m\right) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N}\varphi(t)e_0(t)$$

and apply the Central Limit Theorem. The distribution of Y_N, i.e. $-\sqrt{N}V'_N(\theta_0, Z^N)$, converges to a Gaussian distribution as N tends to infinity.

$$Y_N = -\sqrt{N}V'_N(\theta_0, Z^N) \sim N(0, \lambda_0\bar{R}) \tag{17}$$

Next, compute $V''_N(\xi_N, Z^N)$

$$\begin{aligned} V''_N(\xi_N, Z^N) &= \frac{d}{d\theta}V'_N\left(\theta, Z^N\right)\Big|_{\theta=\xi_N} \\ &= \frac{d}{d\theta}\frac{1}{N}\sum_{t=1}^{N}\varepsilon(t,\theta)\frac{d\varepsilon}{d\theta}\Big|_{\theta=\xi_N} \\ &= \frac{1}{N}\sum_{t=1}^{N}\left\{\frac{d\varepsilon}{d\theta}\left(\frac{d\varepsilon}{d\theta}\right)^T + \varepsilon(t,\theta)\underbrace{\frac{d^2\varepsilon}{d\theta^2}}_{0}\right\}\Bigg|_{\theta=\xi_N} \\ &= \frac{1}{N}\sum_{t=1}^{N}\left(\phi(t)\phi^T(t)\right) \end{aligned} \tag{18}$$

Therefore, under the ergodicity assumption,

$$\bar{V}''_N(\xi_N, Z^N) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}\left(\varphi(t)\varphi^T(t)\right) = \bar{R} \tag{19}$$

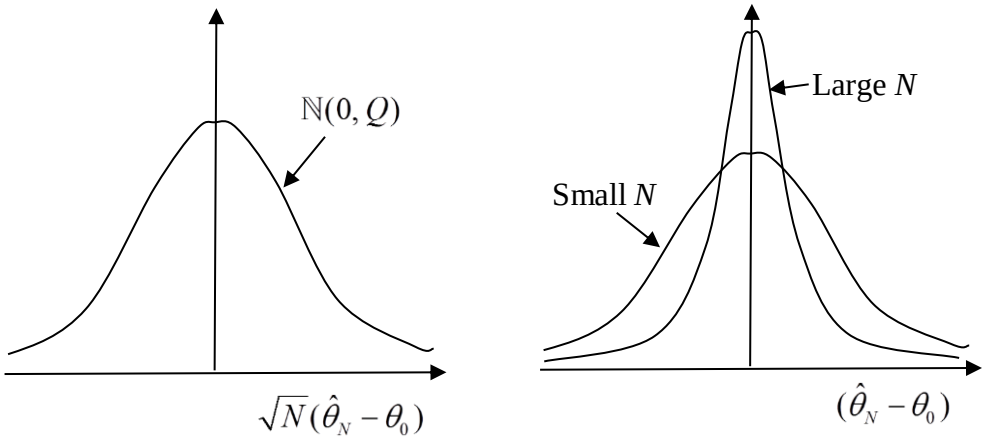
From (10), (17) and (19), the distribution of $\sqrt{N}(\hat{\theta}_N - \theta_0)$ converges to the Gaussian distribution given by

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \sim \mathbb{N}(0, Q)\, as \quad N \to \infty \tag{20}$$

where

$$Q = \bar{R}^{-1}(\lambda_0 \bar{R})\bar{R}^{-1} = \lambda_0 \bar{R}^{-1}$$

Note that, as coordinate transformation $y=Ax$ is performed, the covariant matrix C associated with a multivariate Gaussian distribution is transformed to ACA^T. This is used in (21).



Remark 1

Eq.(20) manifests that the standard deviation of $\hat{\theta}_N - \theta_0$ decreases at the rate of $\frac{1}{\sqrt{N}}$ for large N. See the figure above. Note that $\operatorname{cov}\hat{\theta}_N = \frac{1}{N}Q$.

Confidence Interval and Confidence Level

The above asymptotic variance analysis provides a useful guideline for selecting the number of data points: Confidence Interval and Confidence Level.

For simplicity, let us consider a scalar case, $d = 1$. The covariance is simply given by σ^2. The figure below shows the standard normal distribution where the horizontal axis

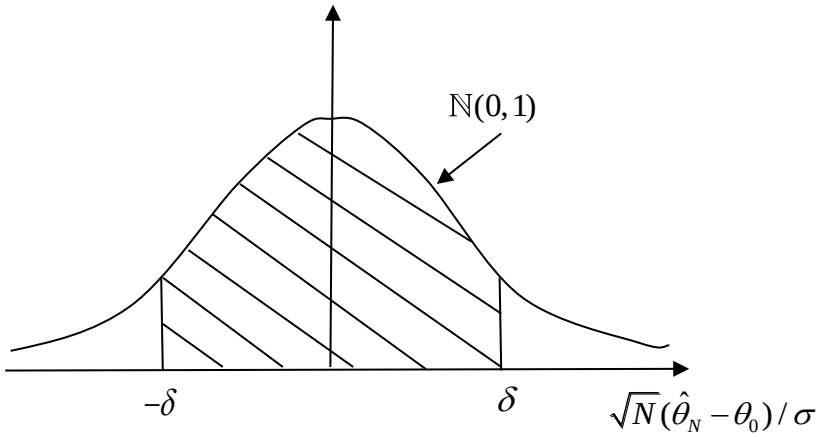
is normalized by the standard deviation σ. The probability that the true value θ_0 is involved in the interval $[-\delta, +\delta]$ is expressed as

$$\Pr\left[\frac{\sqrt{N}}{\sigma}\left|\hat{\theta}_N - \theta_0\right| \le \delta\right] = \Pr\left[\left|\hat{\theta}_N - \theta_0\right| \le \frac{\sigma}{\sqrt{N}}\delta\right] = 1-\alpha \tag{21}$$

1-α is called Confidence Level and

$$\hat{\theta}_N - \frac{\sigma}{\sqrt{N}}\delta \le \theta_0 \le \hat{\theta}_N + \frac{\sigma}{\sqrt{N}}\delta$$

is called Confidence Interval.



Remark 2

The above result is for a very restrictive case. A similar result can be obtained for general cases with mild assumptions.

- The true system (7) does not have to be assumed. Instead, $\theta^* = \arg\min \bar{V}(\theta)$ must be involved in D_M.
- The linear regression (6) can be extended to a general predictor where the model parameter θ is determined based on the prediction error method (4), (5).

The extended result of estimate distribution is summarized in the following theorem, i.e. Ljung's Textbook Theorem 9-1.

Theorem: Asymptotic Variance Consider the estimate $\hat{\theta}_N$ determined by (4) and (5). Assume that the model structure is linear and uniformly stable and that the data set Z^∞ satisfies the quasi stationary and ergodicity requirements. Assume also that $\hat{\theta}_N$ converges with probability 1 to a unique parameter vector θ^* involved in D_M:

$$\hat{\theta}_N \to \theta^* \in D_M \qquad w.p.1 \quad as\, N \to \infty \tag{22}$$

and that

$$\bar{V}''_N(\theta^*) > 0 \ ; \ \text{positive definite} \tag{23}$$

and that

$$\sqrt{N} \cdot E\left[\frac{1}{N} \sum_{t=1}^{N} \left\{ \left(\frac{d}{d\theta} \hat{y}(t|\theta) \right) \varepsilon(t,\theta)\Big|_{\theta^*} - m_t \right\} \right] \to 0, \text{ as } N \text{ tends to } \infty \tag{24}$$

where m_t is the ensemble mean given by

$$m_t = \bar{E}\left[\sum_{t=1}^{N} \left(\frac{d}{d\theta} \hat{y}(t|\theta) \right) \varepsilon(t,\theta)\Big|_{\theta^*} \right] \tag{25}$$

Then, the distribution of $\sqrt{N}(\hat{\theta}_N - \theta_0)$ converges to the Gaussian distribution given by

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \sim N(0, P_\theta) \tag{26}$$

where P_θ is given by

$$P_\theta = \left[\bar{V}''_N(\theta^*) \right]^{-1} Q \left[\bar{V}''_N(\theta^*) \right]^{-1} \tag{27}$$

$$Q = \lim_{N \to \infty} N \cdot E[\left(\bar{V}'_N(\theta^*) \right) \left(\bar{V}'_N(\theta^*) \right)^T] \tag{28}$$

The proof is quite complicated, since the random variable $\left(\frac{d}{d\theta} \hat{y}(t|\theta) \right) \varepsilon(t,\theta)\Big|_{\theta^*}$ is not independent. Therefore, the standard central limit theorem is not applicable.

Appendix 9A, at p.309 of Ljung's textbook, shows the outline of proof. Since the model structure is assumed to be stable uniformly in θ, X_t and X_s are independent as t and s are distal. Because of this property, the sum, $\frac{1}{\sqrt{N}} \sum_{t=1}^{N} (X_t - m_t)$, converges to the Gaussian distribution.

14.4 Expression for the Asymptotic Variance.

As stated formally in Theorem 5, the distribution of $\sqrt{N}\left(\hat{\theta}_N - \theta^*\right)$ converges to a Gaussian distribution for the broad class of system identification problems. This implies that the covariance of $\hat{\theta}_N$ asymptotically converges to:

$$Cov\hat{\theta}_N \sim \frac{1}{N} P_\theta \tag{29}$$

This is called the asymptotic covariance matrix.

The asymptotic variance depends not only on

(a) the member of samples/data set size: *N*, but also on

(b) the parameter sensitivity of the predictor:

$$\psi(t,\theta^*) = \frac{d}{d\theta}\hat{y}(t|\theta)\Big|_{\theta^*} = -\frac{d}{d\theta}\varepsilon(t,\theta)\Big|_{\theta^*} \text{ and} \tag{30}$$

(c) Noise variance λ_0.

Let us compute the covariance once again for the general case. From (5) and (30),

$$\frac{d}{d\theta}V_N(\theta, Z^N) = \frac{1}{N}\sum_{t=1}^{N}\varepsilon(t,\theta)\frac{d\varepsilon}{d\theta} = -\frac{1}{N}\sum_{t=1}^{N}\varepsilon(t,\theta)\psi(t,\theta) \tag{31}$$

Unlike the linear regression, the sensitivity $\psi(t,\theta)$is a function of θ,

$$\begin{aligned}\frac{d^2}{d\theta^2}V_N(\theta, Z^N) &= -\frac{1}{N}\sum\left(\frac{d\varepsilon}{d\theta}\psi + \varepsilon\frac{d\psi}{d\theta}\right)\\ &= \frac{1}{N}\sum_{t=1}^{N}\left(\psi(t,\theta)\psi^T(t,\theta) - \varepsilon(t,\theta)\frac{d^2}{d\theta^2}\hat{y}(t|\theta)\right)\end{aligned} \tag{32}$$

When the true system is contained in the model structure, $\theta_0 \in D_M$, and that is unique,

$$\varepsilon(t,\theta_0) = e_0(t) \tag{33}$$

from (28), (31), and (33)

$$\begin{aligned}Q &= \lim_{N\to\infty}\frac{N}{N^2}\sum_{t=1}^{N}\sum_{s=1}^{N}E\left[e_0(t)\psi(t,\theta_0)\psi^T(s,\theta_0)e_0(s)\right]\\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\lambda_0 E\left[\psi(t,\theta_0)\psi^T(t,\theta_0)\right] = \lambda_0\bar{E}\left[\psi(t,\theta_0)\psi^T(t,\theta_0)\right]\end{aligned} \tag{34}$$

Also from (32)

$$\bar{V}''(\theta_0) = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\left[E\left[\psi(t,\theta_0)\psi^T(t,\theta_0)\right] - \varepsilon(t,\theta_0)\frac{d^2}{d\theta^2}\hat{y}(t|\theta)\Big|_{\theta_0}\right] \tag{35}$$

$$= \bar{E}\left[\psi(t,\theta_0)\psi^T(t,\theta_0)\right] - \bar{E}\left[e_0(t)\frac{d^2}{d\theta^2}\hat{y}\Big|_{\theta_0}\right]$$

0

This depends on Z^{t-1} not on Z^t

Since $e_0(t)$ and $\frac{d^2}{d\theta^2}\hat{y}$ are independent, the second term varnishes. Substituting (34) and (35) into (29),

$$Cov\hat{\theta}_N \sim \frac{1}{N}P_\theta = \frac{\lambda_0}{N}\left[\bar{E}\left(\psi(t,\theta_0)\psi^T(t,\theta_0)\right)\right]^{-1} \tag{36}$$

The asymptotic variance is therefore a) inversely proportional to the number of samples, b) proportional to the noise variance, and c) inversely related to the parameter sensitivity. The more a parameter affects the prediction, the smaller the variance becomes.

Since θ_0 is not known, the asymptotic variance cannot be determined. In practice, however, an empirical estimate, like the following formula, works well for large N.

$$\hat{P}_N = \hat{\lambda}_N\left[\frac{1}{N}\sum_{t=1}^{N}\psi(t,\hat{\theta}_N)\psi^T(t,\hat{\theta}_N)\right]^{-1} \tag{37}$$

$$\hat{\lambda}_N = \frac{1}{N}\sum_{t=1}^{N}\varepsilon^2(t,\hat{\theta}_N) \tag{38}$$

If one computes $\hat{P}_N$ during experiments, sufficient data samples needed for assuring the model accuracy may be obtained.